Maximum Matching and Linear Programming in Fixed-Point Logic with Counting

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Is there is logical characterisation of P on unordered structures?

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We show:

- $\textsc{MaximumMatching} \in$ FPC.

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We show:

- MaximumMatching $\in$ FPC.
- LinearProgramming $\in$ FPC.

Structures, Numbers, and Matrices

Vocabulary τ

Finite τ-structures $\mathrm{fin}[\tau]$

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- $\tau_{\mathbb{Q}}$ Encodes the binary expansion of a rational number $q \in \mathbb{Q}$ in a domain of ordered bits B.
- τ_{vec} Encodes a vector $v \in \mathbb{Q}^I$ as a set of rational numbers indexed by a separate domain I.
- τ_{mat} Encodes a matrix $M \in \mathbb{Q}^{I \times J}$ as a set of rational numbers indexed by a pair of separate domains I and J.

FPC and Interpretations

FPC   Inflationary fixed-point logic extended with the ability to express the size of definable sets.

- Assume standard syntax and semantics.
- FPC$[\tau]$ defines relations over $\mathrm{dom}(\mathcal{A}) \uplus [|\mathrm{dom}(\mathcal{A})| + 1]$ invariant to automorphisms of $\mathcal{A} \in \mathrm{fin}[\tau]$.

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FPC interpretations can express many standard linear algebraic operations, e.g., multiplication, inverse, and rank [Holm '10].

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Consider the Euclidean space $\mathbb{Q}^V$ indexed by a set V.

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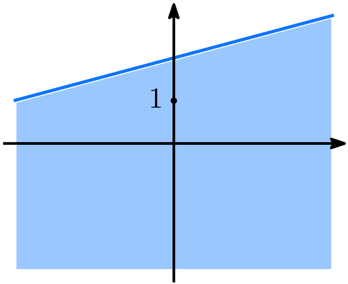

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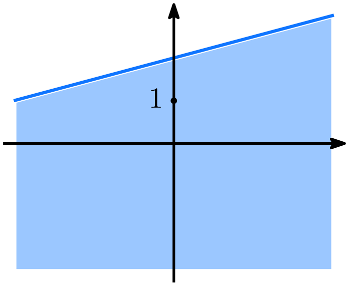

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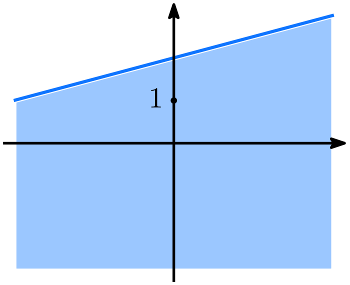

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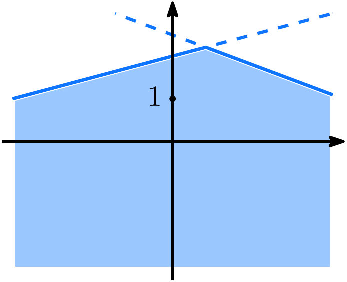

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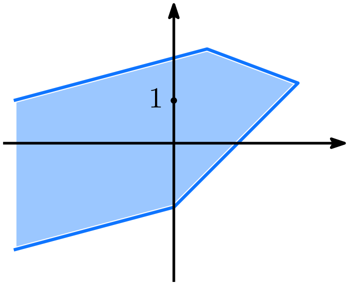

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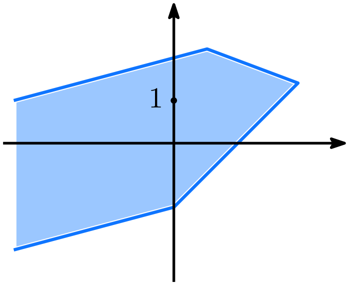

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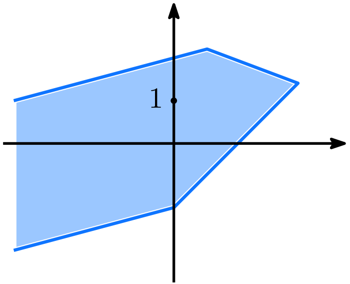

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Linear Optimisation Problem

Given: A polytope $P \subseteq \mathbb{Q}^V$ and objective vector $k \in \mathbb{Q}^V$.
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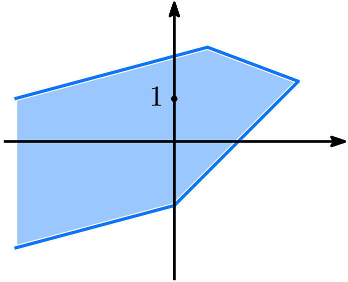

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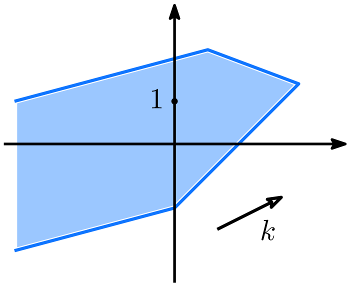

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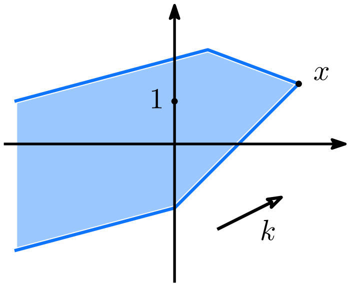

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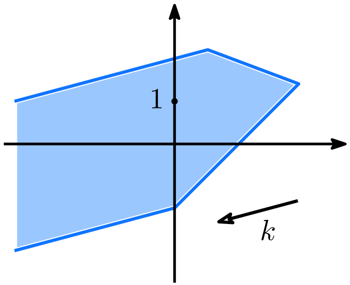

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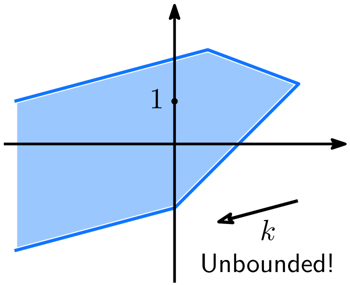

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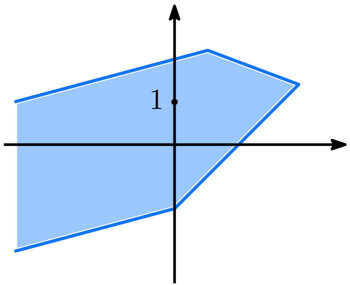

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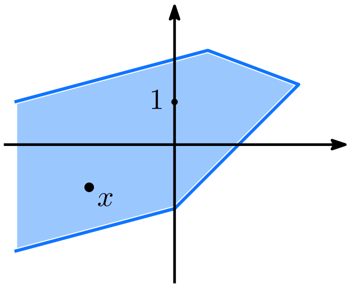

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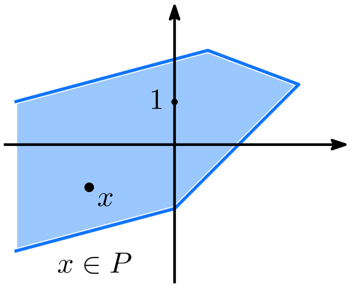

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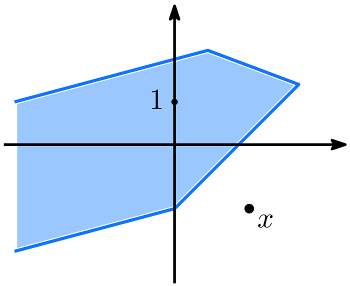

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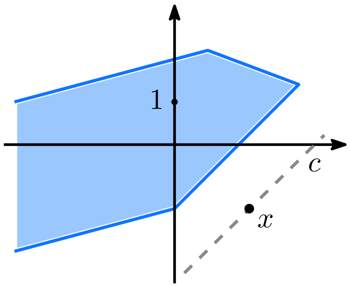

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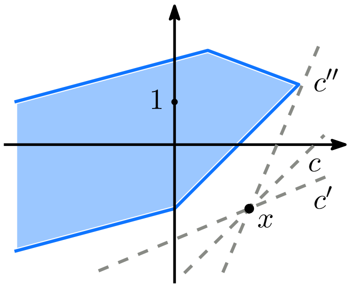

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For polytopes in ordered spaces, the separation and optimisation problem are polynomial-time equivalent (via the ellipsoid method [Khachiyan '79]).

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Typical algorithm for solving separation on explicit polytope $P_{A,b}$.

$\mathrm{SEP}(A \in \mathbb{Q}^{C \times V}, b \in \mathbb{Q}^C, x \in \mathbb{Q}^V)$:

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$\Rightarrow$

1. If $Ax \leq b$, return "$x \in P$".
2. $c \leftarrow \sum_{\{r \in C \mid A_r x > b_r\}} A_r$.
3. If $c = 0^V$, return 1^V.
4. Return c.

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Representation

A representation (τ, ν) of a class $\mathcal{P}$ of polytopes is

- a vocabulary τ, and
- an onto function $\nu : \mathrm{fin}[\tau] \to \mathcal{P}$ which is isomorphism invariant, i.e., $\mathcal{A} \cong \mathcal{B} \Rightarrow \nu(\mathcal{A}) \cong \nu(\mathcal{B})$, $\forall \mathcal{A}, \mathcal{B} \in \mathrm{fin}[\tau]$.

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- an onto function $\nu : \mathrm{fin}[\tau] \to \mathcal{P}$ which is isomorphism invariant, i.e., $\mathcal{A} \cong \mathcal{B} \Rightarrow \nu(\mathcal{A}) \cong \nu(\mathcal{B})$, $\forall \mathcal{A}, \mathcal{B} \in \mathrm{fin}[\tau]$.

Explicit representation takes $\mathrm{fin}[\tau_{\mathsf{mat}} \uplus \tau_{\mathsf{vec}}]$ to the class of all polytopes via $\nu : (A, b) \mapsto P_{A,b}$.

A representation (τ, ν) is well described if for all $\mathcal{A} \in \mathrm{fin}[\tau]$, $\langle \nu(\mathcal{A}) \rangle = \mathrm{poly}(|\mathcal{A}|)$.

- The explicit representation is trivially well described.
- There are well-described representations with an exponential number of constraints.

Convex Optimisation – Representation, contd.

Expressing Linear Optimisation

Let $\mathcal{P}_{\tau,\nu}$ be a class of polytopes given by a representation (τ, ν). The linear optimisation problem for $\mathcal{P}_{\tau,\nu}$ is expressible in FPC if there is an FPC interpretation

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An analogous definition can be made for the separation problem.

Main Result – Linear Programming $\in$ FPC

Theorem (c.f., e.g., [GLS88, Theorem 6.4.9])

Let $\mathcal{P}$ be a class of well-described polytopes. Then,

$$\text{linear optimisation on } \mathcal{P} \leq^{p}_{T} \text{ separation on } \mathcal{P}$$

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Corollary (Linear Programming $\in$ FPC)

There is an FPC *interpretation expressing the linear optimisation problem w.r.t. the explicit representation.*

Main Result – Maximum Matching $\in$ FPC

Maximum Matching Problem

Given: A graph $G = (V, E)$ by an incidence matrix $\{0,1\}^{V \times E}$.
Determine: $M \subseteq E$ such that

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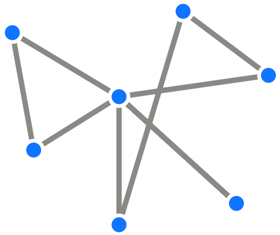

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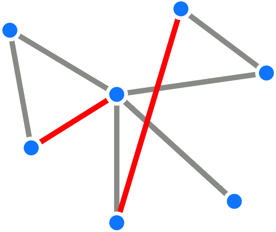

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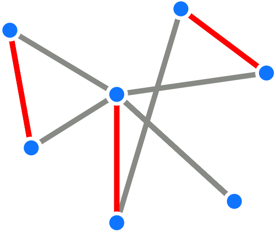

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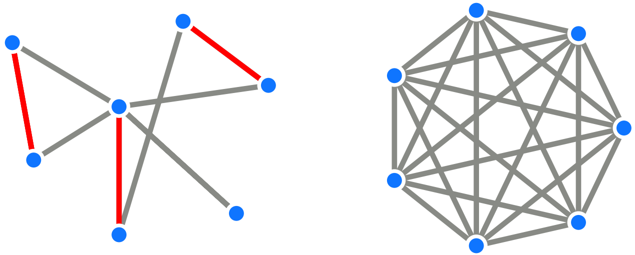

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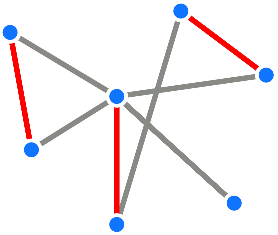

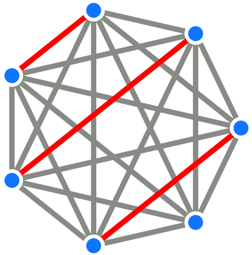

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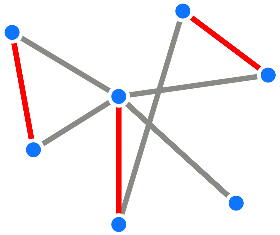

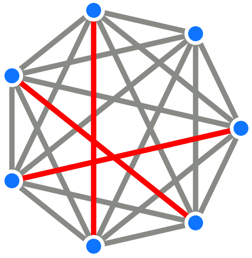

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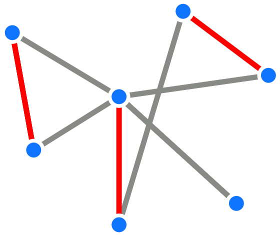

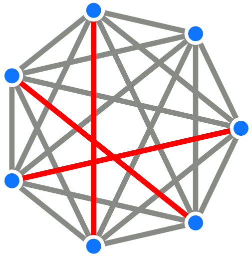

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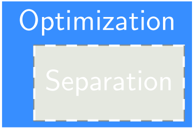
This answers an open question of [Blass-Gurevich-Shelah '99].

Proof Overview



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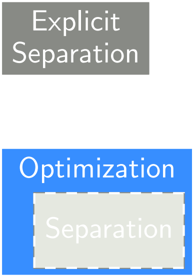

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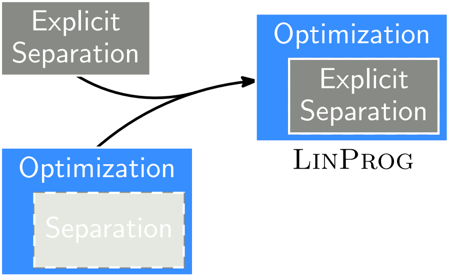

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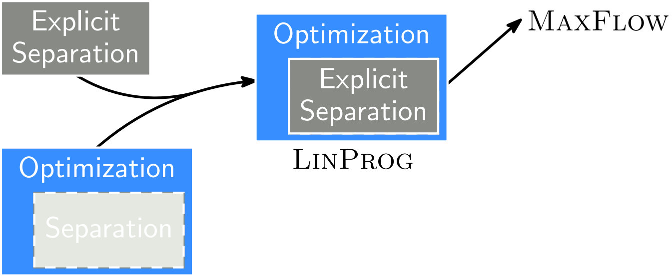

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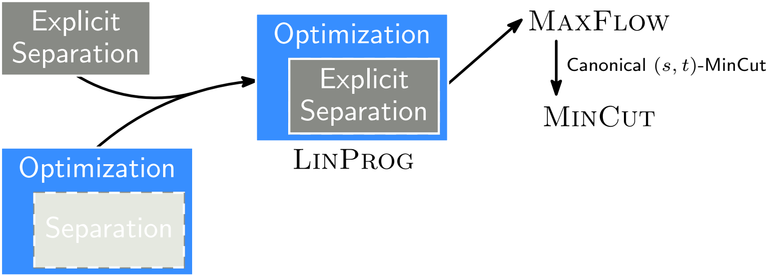

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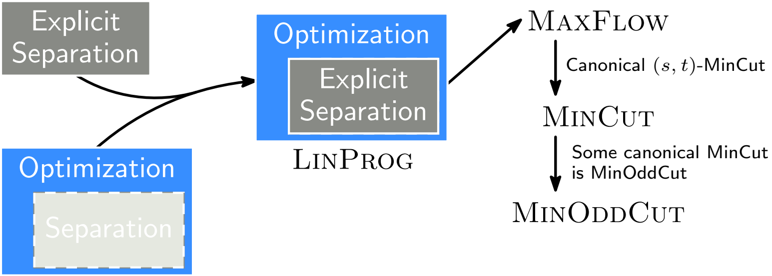

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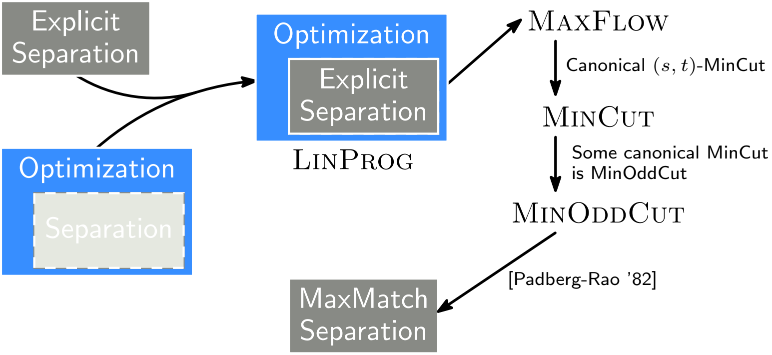

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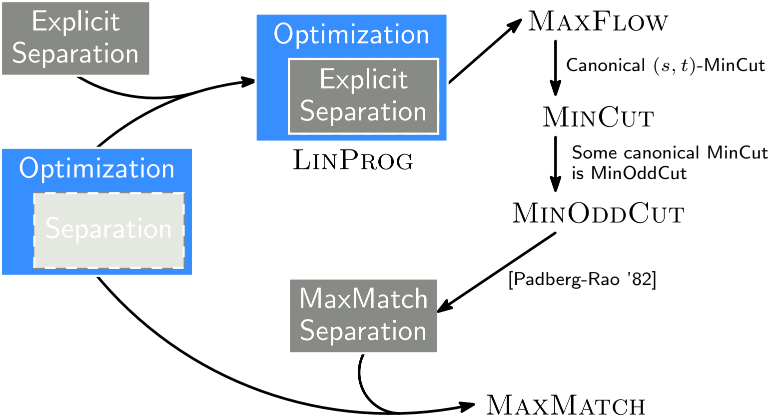

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We can use such c to construct an approximate σ.

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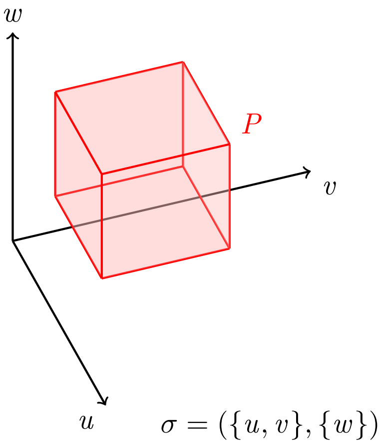

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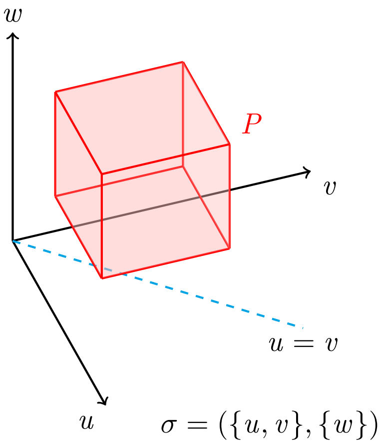

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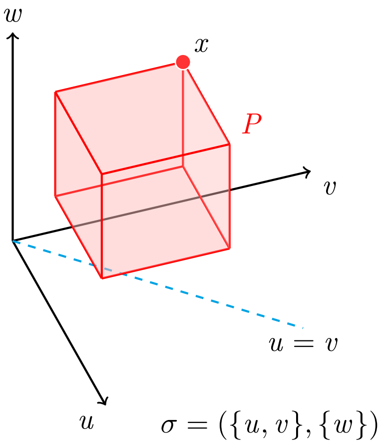

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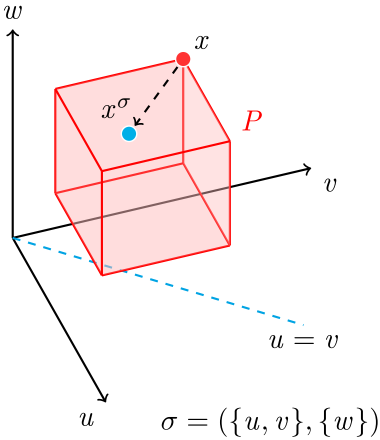

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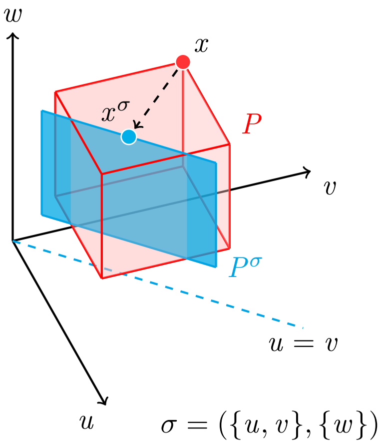

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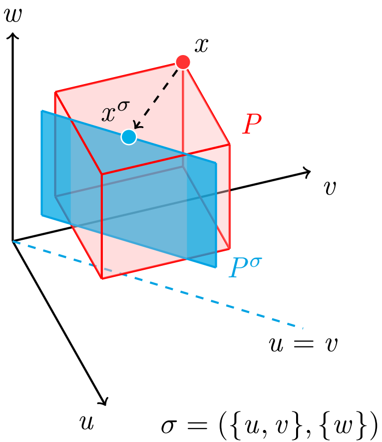

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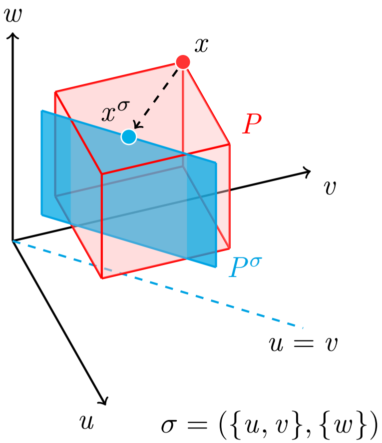

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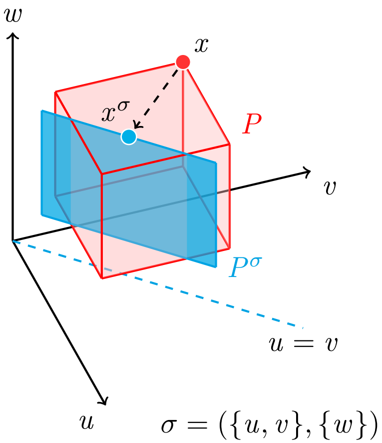
- An optimum of P^σ gives an optimum of P.
- $\mathrm{SEP}(P^\sigma, x)$ reduces to $\mathrm{SEP}(P, x^{-\sigma}) = c$, but... only if c agrees with σ.



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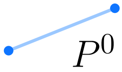

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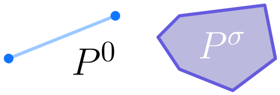

Proof Sketch – Optimisation to Separation, contd.

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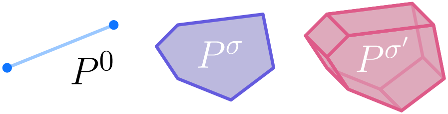

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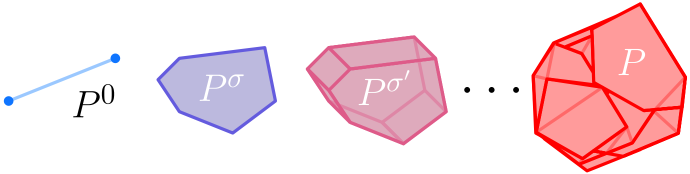

Summary

Prove FPC analog of P-reduction from optimisation to separation.

Theorem (Main)

Let $\mathcal{P}_{\tau,\nu}$ be a class of well-describe polytopes given by τ-structures and the function ν. Then,

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And use it to prove several optimisation problems are in FPC.

Theorem

The follow problems are expressible in FPC*:*

- LinProg,
- MaxFlow / MinCut,
- MinOddCut, *and*
- MaxMatch.

Open Questions

- Extend our main reduction to:
  - quadratic programs,
  - semidefinite programs, or
  - convex programs.
- What other problems can be put in FPC?
- Is linear programming complete for FPC under FO interpretations?
- Do our results provide a route to proving integrality gaps for hierarchies of linear programming relaxations using inexpressibility in FPC?

Thanks!